

Confined System with Rashba Coupling in Constant Magnetic Field

Mohammed El Bouziani^a, Rachid Houça^a and Ahmed Jellal^{a,b,c*}

^a*Theoretical Physics Group, Faculty of Sciences, Chouaib Doukkali University,
PO Box 20, 24000 El Jadida, Morocco*

^b*Physics Department, College of Science, King Faisal University,
PO Box 380, Alahsa 31982, Saudi Arabia*

^c*Saudi Center for Theoretical Physics, Dhahran, Saudi Arabia*

Abstract

We study a two dimensional system of electrons with Rashba coupling in the constant magnetic field B and confining potential. We algebraically diagonalize the corresponding Hamiltonian to end up with the solutions of the energy spectrum. In terms of two kinds of operator we construct two symmetries and discuss the filling of the shells with electrons for strong and weak B . Subsequently, we show that our system is sharing some common features with quantum optics where the exact operator solutions for the basics Jaynes-Cummings variables are derived from our results. An interesting limit is studied and the corresponding quantum dynamics is recovered.

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*ajellal@ictp.it – a.jellal@ucd.ac.ma

1 Introduction

The spin-orbit coupling, which couples the electron spin and its orbital motion, has been the subject of several theoretical and experimental research [1]. This opens the door to developing a new generation of electronic spin (spintronics) and presents a new branch of physics of semiconductors. It puts the spin of the electron at the center of interest and exploits the spin-dependent electronic properties of magnetic materials and semiconductors. The underlying basis for this new electronics is the intimate connection between the electron charge and spin. A crucial implication of this relationship is that one can have access to spin through the spin property of the electron orbital in the solid. The link between the electron charge and spin is expressed by the spin-orbit interaction in semiconductors.

Novel spin properties arise from the interplay between Rashba spin splitting [2] and further confinement of two dimensional (2D) electrons in quantum wires [3, 4, 5, 6], rings [7, 8] or dots [9, 10, 11, 12, 13, 14, 15]. Spin-orbit coupling has also been shown to affect the statistics of energy levels and eigenfunctions as well as current distributions [16, 17]. The interplay between spin-orbit coupling and external magnetic fields was analyzed theoretically using random matrix theory [18]. With this respect, Schliemann [19] studied the cyclotron motion and magnetic focusing in semiconductor quantum wells with spin-orbit coupling. More precisely, the ballistic motion of electrons in III-V semiconductor quantum wells with Rashba spin-orbit coupling in a perpendicular magnetic field was investigated. Taking into account the full quantum dynamics of the problem, the modifications of classical cyclotron orbits due to spin-orbit interaction was explored and the analogy to Jaynes-Cummings was established.

On the other hand, Jaynes-Cummings model is describing the basic interaction of a two-level atom and quantized field, which is also the cornerstone for the treatment of the interaction between light and matter in quantum optics [20]. It can be used to explain many quantum phenomena, such as the collapses and revivals of the atomic population inversions, squeezing of the quantized field and the atom-cavity entanglement. Recent experiments showed that Jaynes-Cummings model can be implicated in quantum-state engineering and quantum information processing, e.g. generation of Fock states [21] and entangled states [22], and the implementations of quantum logic gates [23], etc. Originally, Jaynes-Cummings model is physically implemented with a cavity quantum electrodynamics system, see for instance [24]. Certainly, there has been also interest to realize Jaynes-Cummings model with other physical systems. A typical system is a cold ion trapped in a Paul trap and driven by classical laser beams [25, 26] where the interaction between two selected internal electronic levels and the external vibrational mode of the ion can be induced.

Motivated by different investigations cited above and in particular [19, 27], we develop our proposal to deal with different issues. For this, we consider a 2D system in the presence of an external magnetic field B and study the quantum dynamics. But, we include the parabolic potential to confine our system and Rashba interaction to make contact with quantum optics. Through the Weyl-Heisenberg symmetries, we obtain the solutions of the energy spectrum and construct the algebra $su(2)$ as well as $su(1, 1)$. By considering strong and weak B , we show that our system reduces to the Landau problem for the first case. By using the Heisenberg picture, we derive two copies of the Jaynes-Cummings model oscillating with different frequencies. Finally, we recover the results of without confinement case [19] in simply way to conclude that our findings are general and deserve different extensions.

The present paper is organized as follows. In section 2 we formulate our problem by setting the Hamiltonian and choosing the convenient gauge. In section 3, we introduce a series of annihilation and creation operators to diagonalize our Hamiltonian, which serves to determine explicitly the exact eigenvalues and eigenstates. We construct two symmetries and analyze the system behavior by distinguishing the strong and weak magnetic field cases in section 4. We establish a link with Jaynes-Cummings model and make different comments in section 5. Moreover, to show the relevance of our results we study a limiting case. Finally, we close by concluding our work and giving some perspective.

2 Solutions of energy spectrum

We start by formulating our problem to end up with the appropriate Hamiltonian describing the system under consideration. Subsequently, we use the algebraic approach to determine explicitly the eigenvalues as well as the eigenstates. These will be used to discuss the possibility to fill the shells with electrons when the magnetic field is strong and weak.

2.1 Hamiltonian formalism

We consider a system of electrons in the presence of a constant magnetic field $\vec{B} = B\vec{e}_z$ and confining potential. By taking into account of the Rashba spin-orbit coupling and Zeeman effect, the Hamiltonian for a single electron reads as

$$H = \frac{\vec{\pi}^2}{2m} + \frac{1}{2}m\omega_0^2(x^2 + y^2) + \lambda(\pi_x\sigma_y - \pi_y\sigma_x) + \frac{1}{2}g\mu_B B\sigma_z \quad (1)$$

where $\vec{\pi} = \vec{p} + \frac{e}{c}\vec{A}$ is the conjugate momentum and \vec{A} is the vector potential. λ is the Rashba coupling parameter, g is the Landé-factor, μ_B is the Bohr magneton and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. To proceed further, we choose the symmetric Landau gauge

$$\vec{A} = \frac{B}{2}(-y, x, 0) \quad (2)$$

and write the Hamiltonian (1) as

$$\begin{aligned} H = & \frac{1}{2m} \left[\left(p_x - \frac{eB}{2c}y \right)^2 + \left(p_y + \frac{eB}{2c}x \right)^2 \right] + \frac{1}{2}m\omega_0^2(x^2 + y^2) \\ & + \lambda \left[\sigma_y \left(p_x - \frac{eB}{2c}y \right) - \sigma_x \left(p_y + \frac{eB}{2c}x \right) \right] + \frac{1}{2}g\mu_B B\sigma_z. \end{aligned} \quad (3)$$

The algebraic structure of the above Hamiltonian is easily displayed if we adopt the method of separation of Cartesian variables. This process suggests to decompose (3) into four parts

$$H = H_F + H_R + \frac{1}{2}\omega_c L_z + \frac{1}{2}g\mu_B B\sigma_z \quad (4)$$

such that the free part takes the form

$$H_F = \left(\frac{p_x^2}{2m} + \frac{1}{8m}\omega^2 x^2 \right) + \left(\frac{p_y^2}{2m} + \frac{1}{8m}\omega^2 y^2 \right) \quad (5)$$

and the Rashba coupling in magnetic field is given by

$$H_R = \lambda \left[\sigma_y \left(p_x - \frac{eB}{2c} y \right) - \sigma_x \left(p_y + \frac{eB}{2c} x \right) \right]. \quad (6)$$

where $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency, $L_z = xp_y - yp_x$ is the angular momentum and we have set the new frequency as $\omega = \sqrt{\omega_c^2 + 4\omega_0^2}$. To close this part, we emphasize that (4) is splitting into two independent harmonic oscillator Hamiltonian's supplemented by the angular momentum and Rashba spin-orbit coupling added to Zeeman term. This convenient form of the Hamiltonian will help us to end up with its diagonalization in the simple way.

2.2 Solution through Weyl-Heisenberg symmetries

We introduce the standard machinery and techniques to get the solutions of energy spectrum of the Hamiltonian (4). Instead of directly using the oscillator annihilation operators

$$a_x = \frac{1}{\sqrt{2}} \left(\frac{x}{l_0} + \frac{il_0}{\hbar} p_x \right), \quad a_y = \frac{1}{\sqrt{2}} \left(\frac{y}{l_0} + \frac{il_0}{\hbar} p_y \right) \quad (7)$$

we work with two new ones, which are linear superposition of a_x and a_y , such as

$$a_d = \frac{1}{\sqrt{2}} (a_x - ia_y), \quad a_g = \frac{1}{\sqrt{2}} (a_x + ia_y) \quad (8)$$

where $l_0 = \sqrt{\frac{2\hbar}{m\omega}}$ being the magnetic length. Note that, a_d and a_g are bosonic operators and satisfy the relation commutations

$$[a_d, a_d^\dagger] = 1 = [a_g, a_g^\dagger] \quad (9)$$

and other relations vanish. from the above operators, one can obtain the useful identities for the conjugate momentum

$$\pi_x = \frac{\hbar}{2il_0} \left[l_1 (a_d - a_d^\dagger) + l_2 (a_g - a_g^\dagger) \right], \quad \pi_y = \frac{\hbar}{2l_0} \left[l_1 (a_d + a_d^\dagger) - l_2 (a_g + a_g^\dagger) \right] \quad (10)$$

as well as for the positions

$$x = \frac{l_0}{2} (a_d + a_d^\dagger + a_g + a_g^\dagger), \quad y = \frac{l_0}{2i} (-a_d + a_d^\dagger + a_g - a_g^\dagger) \quad (11)$$

where we have set $l_1 = \left(1 + \frac{l_0^2}{2l^2}\right)$, $l_2 = \left(1 - \frac{l_0^2}{2l^2}\right)$ and $l^2 = \frac{\hbar}{m\omega_c}$. These algebraic structures will play a crucial role in solving different issues and more precisely in diagonalizing different Hamiltonian's entering in the game.

We start by writing the Rashba Hamiltonian (6) in terms of the annihilation and creation operators introduced above. Indeed, we have

$$H_R = H_R^g + H_R^d \quad (12)$$

where these two parts are given by

$$H_R^g = \lambda \sqrt{\frac{m\hbar\omega}{2}} \left(1 - \frac{\omega_c}{\omega}\right) \begin{pmatrix} 0 & a_g \\ a_g^\dagger & 0 \end{pmatrix}, \quad H_R^d = -\lambda \sqrt{\frac{m\hbar\omega}{2}} \left(1 + \frac{\omega_c}{\omega}\right) \begin{pmatrix} 0 & a_d \\ a_d^\dagger & 0 \end{pmatrix}. \quad (13)$$

In the same way, we can diagonalize the free Hamiltonian and angular momentum to finally end up with a new form of the Hamiltonian (4). This is

$$H = \frac{\hbar\omega}{2} (a_d^\dagger a_d + a_g^\dagger a_g + 1) + \frac{\hbar\omega_c}{2} (a_d^\dagger a_d - a_g^\dagger a_g) + H_R^g + H_R^d + \frac{1}{2}g\mu_B B\sigma_z. \quad (14)$$

To determine the solutions of energy spectrum of the above problem, we solve the eigenvalue equation

$$H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (15)$$

which gives the eigenvalues

$$E_{n_d n_g} = \hbar\omega^+ n_d + \hbar\omega^- n_g \pm \frac{1}{2}\hbar\omega^+ \sqrt{8\frac{m\lambda^2}{\hbar\omega} n_d + 1} \pm \frac{1}{2}\hbar\omega^- \sqrt{8\frac{m\lambda^2}{\hbar\omega} n_g + \left(1 + \frac{g\mu_B B}{\hbar\omega^-}\right)^2} \quad (16)$$

where the new frequencies are $\omega^\pm = \frac{1}{2}(\omega \pm \omega_c)$. The corresponding eigenstates read as

$$|n_g, n_d, \sigma\rangle = u_n^\pm |n_g, n_d, \uparrow\rangle + v_n^\pm |n_g - 1, n_d - 1, \downarrow\rangle \quad (17)$$

and we show that the amplitudes parameterizing these states are given by

$$u_n^\pm = \frac{1}{\sqrt{2}} \left(1 \pm \frac{\hbar\omega + g\mu_B B}{\hbar\omega^- \sqrt{8\frac{m\lambda^2}{\hbar\omega} n_g + 1} + \hbar\omega^+ \sqrt{8\frac{m\lambda^2}{\hbar\omega} n_d + \left(1 + \frac{g\mu_B B}{\hbar\omega^+}\right)^2}} \right)^{\frac{1}{2}} \quad (18)$$

$$v_n^\pm = \frac{\pm i}{\sqrt{2}} \left(1 \mp \frac{\hbar\omega + g\mu_B B}{\hbar\omega^- \sqrt{8\frac{m\lambda^2}{\hbar\omega} n_g + 1} + \hbar\omega^+ \sqrt{8\frac{m\lambda^2}{\hbar\omega} n_d + \left(1 + \frac{g\mu_B B}{\hbar\omega^+}\right)^2}} \right)^{\frac{1}{2}}. \quad (19)$$

Having obtained the solutions of the energy spectrum, let us briefly discuss how to recover an interesting case from what we generated so far. Indeed, in studying cyclotron motion and magnetic focusing in semiconductor quantum wells with spin-orbit coupling, Schliemann [19] introduced the Hamiltonian type (1) without the confining potential. Therefore, to recover the corresponding solutions of energy spectrum, we consider the case $\omega = \omega_c$ or $\omega_0 = 0$ in the previous equations. This gives the eigenvalues

$$\varepsilon_{n_d} = \hbar\omega_c n_d \pm \sqrt{2m\lambda^2 \hbar\omega_c n_d + \frac{1}{4}(\hbar\omega_c + g\mu_B B)^2} \quad (20)$$

as well as the eigenstates

$$|n_g, n_d, \sigma\rangle = u_n^\pm |n_g, n_d, \uparrow\rangle + v_n^\pm |n_g - 1, n_d - 1, \downarrow\rangle \quad (21)$$

where the amplitudes are given by

$$u_n^\pm = \left(\frac{1}{2} \pm \frac{\frac{1}{4}(\hbar\omega_c + g\mu_B B)}{\sqrt{2m\lambda^2 \hbar\omega_c n_d + \frac{1}{4}(\hbar\omega_c + g\mu_B B)^2}} \right)^{\frac{1}{2}} \quad (22)$$

$$v_n^\pm = (\pm i) \left(\frac{1}{2} \mp \frac{\frac{1}{4}(\hbar\omega_c + g\mu_B B)}{\sqrt{2m\lambda^2 \hbar\omega_c n_d + \frac{1}{4}(\hbar\omega_c + g\mu_B B)^2}} \right)^{\frac{1}{2}}. \quad (23)$$

We mention that Schliemann [19] used the notation $\alpha = \hbar\lambda$ and the quantum number $n = n_d$. These show clearly that our findings are general and make difference with what obtained in [19]. Certainly, this will play a crucial role in the forthcoming analysis.

2.3 Symmetries and shells

To make comparison with interesting work [27] dealing with confined 2D system in magnetic field, let us introduce two symmetries. These concern the dynamical symmetries $su(2)$ and $su(1, 1)$, which can be realized in terms of the shell operators. We start with $su(2)$ where the corresponding generators can be realized as

$$S_+ = a_d^\dagger a_g, \quad S_- = a_g^\dagger a_d, \quad S_z = \frac{L_z}{2\hbar} \quad (24)$$

which verify the commutation relations

$$[S_+, S_-] = 2S_z, \quad [S_z, S_\pm] = \pm S_\pm. \quad (25)$$

These give invariant Casimir operator

$$\mathcal{C} = \frac{1}{2} (S_+ S_- + S_- S_+) + S_z^2 = \frac{H_F^2}{\hbar^2 \omega^2} - \frac{1}{4}. \quad (26)$$

Therefore, to a fixed value $\mu = (n_d + n_g)/2$ of the operator $\frac{H_F}{\hbar\omega} - \frac{1}{2}$ there corresponds the $(2\mu + 1)$ -dimensional unitary irreducible representation (UIR) of $su(2)$ in which the operator $S_z = \frac{L_z}{2\hbar}$ assumes its spectral values in the range $-\mu \leq \gamma = (n_d - n_g)/2 \leq \mu$.

As far as the second symmetry $su(1, 1)$ is concerned, we consider the generators

$$T_+ = a_d^\dagger a_g^\dagger, \quad T_- = a_g a_d, \quad T_0 = \frac{H_F}{\hbar\omega} \quad (27)$$

satisfying the relations

$$[T_+, T_-] = -2T_0, \quad [T_0, T_\pm] = \pm T_\pm. \quad (28)$$

The Casimir operator then is given by

$$\mathcal{D} = \frac{1}{2} (T_+ T_- + T_- T_+) - T_0^2 = -\frac{1}{4} \left(\frac{L_z^2}{\hbar^2} - 1 \right). \quad (29)$$

Similarly to the previous case, to a fixed value $\eta = (n_d - n_g)/2 + 1/2 \geq 1/2$, with $n_d - n_g = \alpha \geq 0$, of the operator $\frac{1}{2} \left(\frac{L_z}{\hbar} + 1 \right)$ there corresponds a UIR of $su(1, 1)$ in the discrete series, in which the operator $T_0 = \frac{1}{2} \left(\frac{L_z}{\hbar} + 1 \right) + N_g$ assumes its spectral values in the infinite range $\eta, \eta + 1, \eta + 2, \dots$. Alternatively, to a fixed value $\varrho = -(n_d - n_g)/2 + 1/2 \geq 1/2$, with $n_g - n_d = -\alpha \geq 0$ of the operator $\frac{1}{2} \left(-\frac{L_z}{\hbar} + 1 \right)$ there corresponds a UIR of $su(1, 1)$ in the discrete series, in which the operator $T_0 = \frac{1}{2} \left(-\frac{L_z}{\hbar} + 1 \right) + N_d$ assumes its spectral values in the infinite range $\varrho, \varrho + 1, \varrho + 2, \dots$.

Below we will see the importance of the both symmetries introduced above when we analyze two interesting cases. With these we underline the system behavior with respect to different limit of the magnetic field B . We start our analysis by considering the case where B is strong, which is equivalent to the limit $\omega_c \gg \omega_0$, and gives the frequencies $\omega^+ \simeq \omega_c$ and $\omega^- \simeq 0$. These tell us that the total energy can be approximated by

$$E_{n_d} \simeq \hbar\omega_c \left(n_d \pm \frac{1}{2} \right) \pm \frac{1}{2} g\mu_B B \quad (30)$$

where the quantum number is $n_d = 0, 1 \dots$. Now by scaling the energy as

$$E_{n_d} \mp \frac{1}{2}g\mu_B B \simeq \hbar\omega_c \left(n_d \pm \frac{1}{2} \right) = \varepsilon_{n_d} \quad (31)$$

one can realize immediately that our system behaves like a harmonic oscillator in 2D (Landau problem) and for a given n_d there is an infinite degeneracy of the Landau levels. We notice that there are type of quantum numbers $n_d + \frac{1}{2}$ and $n_d - \frac{1}{2}$, which means that we have two Hilbert spaces. However, both spaces are connected via a linear transformation $n = n_d + 1$ that allows to move from one space to another and vice versa. From symmetry point view, this behavior can be regarded as our system has ladder states for a discrete series representations of the algebra $su(1, 1)$ labeled by $\frac{1}{2}(-\alpha \pm 1)$ where $\alpha = n_d - n_g \leq n_d$ for $\alpha \leq 0$.

For weak magnetic field, which corresponds to the limit $\omega_c \ll \omega_0$, we can approximate the frequencies by $\omega^+ = \omega^- \simeq \omega_0$ and therefore write the total energy as

$$E_{n_g, n_d} \simeq \hbar\omega_0 (n_d + n_g \pm 1) \pm \frac{1}{2}g\mu_B B. \quad (32)$$

To interpret this result let us rearrange it as follows

$$E_{n_g, n_d} \mp \frac{1}{2}g\mu_B B \simeq \hbar\omega_0 (2\lambda \pm 1) = \varepsilon_\lambda \quad (33)$$

which shows clearly that our system becomes now invariant under the algebra $su(2)$ and therefore each Landau level has a degeneracy of order $(2\lambda \pm 1)$. Note that, here also we have two UIR of dimensions $(2\lambda + 1)$ and $(2\lambda - 1)$ for the same algebra where the transition between them can be obtained by defining $\rho = \lambda - 1$. Furthermore, from (32) we see that we need $2(\lambda_0 \pm 1)(2\lambda_0 \pm 1)$ to fill the shells up to the value λ_0 .

3 Link with Jaynes-Cummings model

Very recently, it appeared beautiful connections between different areas of physics. Among them, we cite the extraordinary bridge between condensed matter physics and high energy physics through the link of graphene with quantum electromagnetic [29]. Also, another connection between massless Dirac electrons and quantum optics has been established [30]. These links push and motivate to look for bridges and establish contacts between different systems. For this purpose, we make contact with another area of physics by showing how our system can be linked to quantum optics through a mapping between the corresponding Hamiltonian and Jaynes-Communing model. This may help to strength a good knowledge of different aspects of quantum optics.

3.1 Equivalence between models

To show the relevance of the results obtained so far, we study the presence and absence of the confining potential cases in the Hamiltonian system. For first one, we will show that our Hamiltonian (3) is formally equivalent to two copies of the Jaynes-Cummings models for atomic transition in a radiation field, but oscillating with different frequencies. To do our job, we adopt the same method used by Ackerhalt and Rzazewski [28] in analyzing the operator perturbation theory in the Heisenberg picture. For second one, we derive the corresponding results in simple way from our findings.

To proceed further, we need to rearrange our Hamiltonian in order to deal with each part separately and establish the associate link. For this, we start by splitting (14) into two parts

$$H = H_g + H_d \quad (34)$$

where the first one is

$$H_g = \hbar\omega^- a_g^\dagger a_g + \frac{\hbar\omega}{4} + H_R^g + \frac{1}{4}g\mu_B B\sigma_z \quad (35)$$

and the second reads as

$$H_d = \hbar\omega^+ a_d^\dagger a_d + \frac{\hbar\omega}{4} + H_R^d + \frac{1}{4}g\mu_B B\sigma_z. \quad (36)$$

Clearly, these two part are completely different because they are involving different frequencies and therefore different oscillations.

Let us consider the Hamiltonian (35) and make contact with Jaynes-Cummings model. In doing so, we show that (35) can be written as

$$H_g = \hbar\omega^- M_g - \gamma^- \sigma_z + \zeta(\omega) (a_g^\dagger \sigma^- + a_g \sigma^+) \quad (37)$$

where the two operators M_g and σ^\pm are give by

$$M_g = N_g + \sigma^+ \sigma^- + \frac{\omega_c}{2(\omega - \omega_c)} \mathbb{I} \quad (38)$$

$$\sigma^\pm = \frac{1}{2} (\sigma_x \pm i\sigma_y) \quad (39)$$

and we have set the constants γ^- and $\zeta(\omega)$ as

$$\gamma^- = \frac{1}{4} (2\hbar\omega^- - g\mu_B B) \quad (40)$$

$$\zeta(\omega) = \lambda \sqrt{\frac{m\hbar\omega}{2}} \left(1 - \frac{\omega_c}{\omega}\right). \quad (41)$$

For later use, it convenient to define an operator as

$$C_g = -\gamma^- \sigma_z + \zeta(\omega) (a_g^\dagger \sigma^- + a_g \sigma^+) \quad (42)$$

which verifies the commutation relation $[M_g, C_g] = 0$. It tells us that there are two constants of motion corresponding to the Hamiltonian (37). This will help in studying different dynamics of the involved operators.

To study the dynamics related to the Hamiltonian (37), we introduce the Heisenberg equation of motion for the operators a_g^\dagger and σ^+ . These are

$$\frac{d}{dt} a_g^\dagger = \frac{i}{\hbar} [H_g, a_g^\dagger] \quad (43)$$

$$\frac{d}{dt} \sigma^+ = \frac{i}{\hbar} [H_g, \sigma^+]. \quad (44)$$

A straightforward calculation leads

$$\left(i\hbar \frac{d}{dt} + \hbar\omega^-\right) a_g^\dagger = -\zeta(\omega) \sigma^+ \quad (45)$$

$$\left(i\hbar \frac{d}{dt} + \hbar\omega^- - 2\gamma^-\right) \sigma^+ = \zeta(\omega) a_g^\dagger \sigma_z. \quad (46)$$

Using the relations $\sigma^+ \sigma^+ = \sigma^- \sigma^- = 0$, $\sigma^+ \sigma_z = -\sigma^+$ and $\sigma^- \sigma_z = \sigma^-$, we show that (46) can be written in terms of the constant of motion C_g as

$$\left(i\hbar \frac{d}{dt} + \hbar\omega^- + 2C_g\right) \sigma^+ = \zeta(\omega) a_g^\dagger. \quad (47)$$

From (45) and (47), we derive the same second order differential equation for $\sigma^+(t)$ and $a_g^\dagger(t)$. This is given by

$$\left(i\hbar \frac{d}{dt} + \hbar\omega^-\right) \left(i\hbar \frac{d}{dt} + \hbar\omega^- + 2C_g\right) \begin{pmatrix} \sigma^+ \\ a_g^\dagger \end{pmatrix} = -\zeta^2(\omega) \begin{pmatrix} \sigma^+ \\ a_g^\dagger \end{pmatrix}. \quad (48)$$

We are looking for the quantum dynamics, then we need to solve (45) and (47). One way to do is to find solutions of the form

$$\sigma^+(t) = e^{i\beta_g^+ t/\hbar} s_g^+ + e^{i\beta_g^- t/\hbar} s_g^- \quad (49)$$

$$a_g^\dagger(t) = e^{i\beta_g^+ t/\hbar} l_g^+ + e^{i\beta_g^- t/\hbar} l_g^- \quad (50)$$

where β_g^\pm , l_g^\pm , and s_g^\pm are initial time operators. From the shape of the decomposition (45) and (47), one can remark that they have an analogy with the solution of ordinary harmonic oscillator. Therefore, the solution will be of the form $e^{i\beta_g t/\hbar}$ and thus after substitution into (48) gives a second order equation for β_g . This is

$$\beta_g^2 - 2(\hbar\omega^- + C_g) \beta_g + \hbar\omega^- (\hbar\omega^- + 2C_g) + \zeta^2(\omega) = 0. \quad (51)$$

By requiring the condition $[\beta_g, C_g] = 0$, we show that the corresponding solutions under the decomposition forms

$$\beta_g^\pm = \hbar\omega^- + \alpha_g^\pm \quad (52)$$

where α_g^\pm are given by

$$\alpha_g^\pm = \hbar\omega^- + C_g \pm \sqrt{C_g^2 - \zeta^2(\omega)}. \quad (53)$$

From the decomposition of $a_g^\dagger(t)$ and $\sigma^+(t)$, one can notice that the operators constants s_g^\pm and l_g^\pm can be obtained by fixing $t = 0$ in (49) and (50). These give the relations

$$\sigma^+(0) = s_g^+ + s_g^- \quad (54)$$

$$a_g^\dagger(0) = l_g^+ + l_g^-. \quad (55)$$

Injecting the forms (49)-(50) into (45) and (47) to end up with

$$\alpha_g^+ l_g^+ + \alpha_g^- l_g^- = \zeta(\omega) (s_g^+ + s_g^-) \quad (56)$$

$$\alpha_g^- s_g^+ + \alpha_g^+ s_g^- = \zeta(\omega) (l_g^+ + l_g^-). \quad (57)$$

These can be solved to obtain the initial operators as

$$s_g^\pm = \frac{\pm \zeta(\omega) a_g^\dagger(0) \mp \alpha_g^\pm \sigma^+(0)}{\alpha_g^- - \alpha_g^+} \quad (58)$$

$$l_g^\pm = \frac{\mp \zeta(\omega) \sigma^+(0) \pm \alpha_g^\mp a_g^\dagger(0)}{\alpha_g^- - \alpha_g^+}. \quad (59)$$

Combining all to end up with the final solutions of (45) and (47). These are

$$\sigma^+(t) = \frac{e^{i\omega^- t}}{\alpha_g^- - \alpha_g^+} \left\{ \left(\zeta(\omega) a_g^\dagger(0) - \alpha_g^+ \sigma^+(0) \right) e^{i\alpha_g^+ t/\hbar} + \left(-\zeta(\omega) a_g^\dagger(0) + \alpha_g^+ \sigma^+(0) \right) e^{i\alpha_g^- t/\hbar} \right\} \quad (60)$$

$$a_g^\dagger(t) = \frac{e^{i\omega^- t}}{\alpha_g^- - \alpha_g^+} \left\{ \left(-\zeta(\omega) \sigma^+(0) + \alpha_g^- a_g^\dagger(0) \right) e^{i\alpha_g^+ t/\hbar} + \left(\zeta(\omega) \sigma^+(0) - \alpha_g^+ a_g^\dagger(0) \right) e^{i\alpha_g^- t/\hbar} \right\}. \quad (61)$$

They constitute the exact operator solutions for the basic Jaynes-Cummings variables, which have been obtained in [28].

As far as the second part (34) is concerned, we apply the same machinery as before to derive similar results. Indeed, by introducing the operator

$$M_d = N_d + \sigma^+ \sigma^- - \frac{\omega_c}{2(\omega + \omega_c)} \mathbb{I} \quad (62)$$

and the two constants

$$\gamma^+ = \frac{1}{4} (2\hbar\omega^+ - g\mu_B B) \quad (63)$$

$$\zeta'(\omega) = -\lambda \sqrt{\frac{m\hbar\omega}{2}} \left(1 + \frac{\omega_c}{\omega} \right) \quad (64)$$

we write the Hamiltonian H_d (36) as

$$H_d = \hbar\omega^+ M_d - \gamma^+ \sigma_z + \zeta'(\omega) \left(a_d^\dagger \sigma^- + a_d \sigma^+ \right). \quad (65)$$

Doing the same job to find the required solutions

$$\sigma^+(t) = \frac{e^{i\omega^+ t}}{\alpha_d^- - \alpha_d^+} \left\{ \left(\zeta'(\omega) a_d^\dagger(0) - \alpha_d^+ \sigma^+(0) \right) e^{i\alpha_d^+ t/\hbar} + \left(-\zeta'(\omega) a_d^\dagger(0) + \alpha_d^+ \sigma^+(0) \right) e^{i\alpha_d^- t/\hbar} \right\} \quad (66)$$

$$a_d^\dagger(t) = \frac{e^{i\omega^+ t}}{\alpha_d^- - \alpha_d^+} \left\{ \left(-\zeta'(\omega) \sigma^+(0) + \alpha_d^- a_d^\dagger(0) \right) e^{i\alpha_d^+ t/\hbar} + \left(\zeta'(\omega) \sigma^+(0) - \alpha_d^+ a_d^\dagger(0) \right) e^{i\alpha_d^- t/\hbar} \right\} \quad (67)$$

where different quantities are given by

$$\alpha_d^\pm = C_d \pm \sqrt{C_d^2 - \zeta'^2(\omega)} \quad (68)$$

$$C_d = -\gamma^+ \sigma_z + \zeta'(\omega) \left(a_d^\dagger \sigma^- + a_d \sigma^+ \right). \quad (69)$$

These solutions are showing how to obtain the second copy of the Jaynes-Cummings model from our findings. In summary, we conclude that our Hamiltonian is equivalent to two copies of the Jaynes-Cummings model but oscillating with different frequencies.

We close this part by noting that, we can easily obtain the dynamic of the complex position from the above solutions. Indeed, using (11) to write

$$z(t) = l_0 \left(a_d^\dagger(t) + a_g(t) \right) \quad (70)$$

and therefore summing up the adjoint time evolution operator of (61) and (67) to end up with the dynamics of $z(t)$. The established link shows clearly that our system is sharing some common features with quantum optics. Thus, one can use the present system to handle different issues related to Jaynes-Cummings model and vice versa.

3.2 Limiting case

The above results show the analogy to Jaynes-Cummings model and therefore allow us to establish a relation with the already published work [28]. Now, we study the case where the confining potential is absent, which naturally should lead to the Schliemann results [19] for Jaynes-Cummings model. These will be derived in simple way from our findings to show clearly that our work is general and can be extended to deal with different issues.

To recover the dynamics obtained in [19] for Jaynes-Cummings model, we start our job by fixing the frequency as $\omega_0 = 0$ in the previous results. This requirement leads to the constraints

$$\omega^- = 0, \quad l_1 = 2, \quad l_2 = 0 \quad (71)$$

and therefore according to the dynamical equation (45) or (61) we end up with an operator a_g^\dagger time independent

$$a_g^\dagger(t) = a_g^\dagger(0) = a_g^\dagger. \quad (72)$$

However, (67) gives $a_d^\dagger(t)$ time dependent

$$a_d^\dagger(t) = \frac{e^{-i\omega_c t}}{r_d^+ - r_d^-} \left\{ \left(\zeta'(\omega_c) \sigma^+(0) - r_d^- a_d^\dagger(0) \right) e^{-i \frac{r_d^+}{\hbar} t} + \left(-\zeta'(\omega_c) \sigma^+(0) + r_d^+ a_d^\dagger(0) \right) e^{-i \frac{r_d^-}{\hbar} t} \right\} \quad (73)$$

where the roots are

$$r_d^\pm = C_d \pm (C_d^2 + \zeta'^2(\omega_c))^{\frac{1}{2}} \quad (74)$$

and all involved functions are now in terms of the cyclotron frequency ω_c instead of ω . Returning back to the definition of different operators to obtain the time evolution of the position operators in the Heisenberg picture. This simply is

$$x(t) + iy(t) = l_0 \left(a_d^\dagger(t) + a_g \right). \quad (75)$$

Since the operators a_g is time independent, then at $t = 0$ we have

$$x(0) + iy(0) = l_0 a_g \quad (76)$$

and then after replacing, we find

$$x(t) + iy(t) = x(0) + iy(0) + l_0 a_d^\dagger(t). \quad (77)$$

From (10) we can express a_d^\dagger in terms of the conjugate momentum as

$$a_d^\dagger = \frac{l_0}{2i\hbar} (\pi_x + i\pi_y) \quad (78)$$

which leads to the final form of the complex position

$$\begin{aligned} x(t) + iy(t) = & x(0) + iy(0) + i \frac{e^{-i(\omega_c + \frac{r_d^+}{\hbar})t}}{r_d^+ - r_d^-} \left(\frac{r_d^-}{\omega_c} \frac{\pi_x + i\pi_y}{m} - i2\lambda\hbar\sigma^+ \right) \\ & - i \frac{e^{-i(\omega_c + \frac{r_d^-}{\hbar})t}}{r_d^+ - r_d^-} \left(\frac{r_d^+}{\omega_c} \frac{\pi_x + i\pi_y}{m} - i2\lambda\hbar\sigma^+ \right) \end{aligned} \quad (79)$$

where the operators valued r_d^\pm are given by

$$r_d^\pm = C_d \pm (C_d^2 + 2\lambda^2 m \hbar \omega_c)^{\frac{1}{2}}. \quad (80)$$

This nothing but the result obtained by Schliemann [19] in dealing with the same system without confinement. Thus, it really shows that our findings are important as well as general in sense that we can derive other results.

4 Conclusion

We have investigated the basic features of confined two-dimensional system with Rashba spin-orbit interaction in the presence of an external magnetic field B . This latter allowed us to end up with a confining potential along x and y -directions that has been used to deal with different issues. In particular, it has been served to split the corresponding Hamiltonian into two parts. This decomposition was useful in sense that different spectrum are obtained and lead to the total solutions of the energy spectrum. We have shown that those obtained by Schliemann [19] can be derived in the simple way from our solutions.

Using different operators involved in the Hamiltonian, we have realized two dynamical symmetries $su(2)$ and $su(1, 1)$. These together with the strength of magnetic field B allowed us to discuss the filing of the shells with electrons. For strong B , we have concluded that our system behaves like a harmonic oscillators in 2D with an infinite degeneracy of the Landau levels. However, for weak B our system becomes invariant under the algebra $su(2)$ and then each Landau level has an finite degeneracy.

To make contact with quantum optics, we have elaborated a method based on building two Hamiltonian's from the original one. Indeed, by splitting this later into two parts we have shown that it is possible to recover the Jaynes-Cummings model, which is describing a system with two states. This has been done by using the Heisenberg dynamics to find the dynamics of the raising Pauli operator σ^+ and creation operators $(a_d^\dagger, a_g^\dagger)$. After solving different equations, we have ended up with the exact operator solutions for the basic Jaynes-Cummings variables those have been obtained in [28]. To show the validity of our results, we have derived those obtained by Schliemann [19] as particular cases.

The present work can be extended to deal with different issues. For instance, we can use the route used by Schliemann [19] to explicitly study the full quantum dynamics. This is based on expanding the initial state of the system in terms of its eigenstates. Another alternative is to use the obtained results to study different issues related to graphene and spin Hall effect.

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